Mathematics: analysis and approaches

Higher Level

Paper 1

WORKED SOLUTIONS

Date: ____________________

2 hours

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

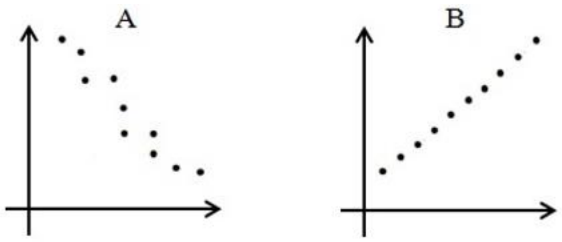
1. [Maximum mark: 6]

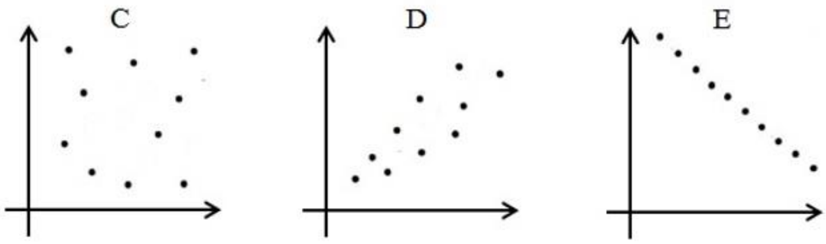
There are seven different plants being studied in a biology class. For each plant, x is the diameter of the stem in centimetres and y is the average leaf length in centimetres. Let r be the Pearson's product-moment correlation coefficient.

(a) Write down the possible minimum and maximum values of r. [2]

(b) Copy and complete the following table by noting which scatter diagram A, B, C, D or E corresponds to each value of r. [4]

correlation coefficient r	scatter diagram
-1	
-0.8	
0	
0.5	





(a) $r_{\text{max}} = 1$, $r_{\text{min}} = -1$

(b)

correlation coefficient r	scatter diagram
-1	**E**
-0.8	**A**
0	**C**
0.5	**D**

2. [Maximum mark: 5]

Let A and B be events such that $\mathrm{P}(A)=0.3$, $\mathrm{P}(B)=0.6$ and $\mathrm{P}(A\cup B)=0.7$. Find $\mathrm{P}(A\mid B)$.

$$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$$

$$0.7=0.3+0.6-\mathrm{P}(A\cap B)$$

$$\mathrm{P}(A\cap B)=0.2$$

$$\mathrm{P}(A\mid B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}=\frac{0.2}{0.6}$$

$$\mathrm{P}(A\mid B)=\frac{1}{3}$$

3. [Maximum mark: 5]

Prove that the sum of the squares of any two consecutive integers is odd.

Let n and $n+1$ be any two consecutive integers where $n \in \mathbb{Z}$

$$n^2 + (n+1)^2 = n^2 + n^2 + 2n + 1$$
$$= 2n^2 + 2n + 1$$
$$= 2(n^2 + n) + 1$$

The expression $2(n^2 + n)$ is divisible by 2, so it must be an even number

Adding 1 to an even number produces an odd number

Hence, the expression $2(n^2 + n) + 1$ must be an odd number

Therefore, the sum of any two consecutive integers is odd ***Q.E.D.***

4. [Maximum mark: 7]

Let $g'(x)=\frac{2x}{\sqrt{3x^2+1}}$. Given that $g(1)=2$, find $g(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)+C\right)=g'(x) \;\Rightarrow\; g(x)+C=\int g'(x)\,\mathrm{d}x$$

$$g(x)+C=\int\frac{2x}{\sqrt{3x^2+1}}\,\mathrm{d}x$$

Let $u=3x^2+1$, then $\mathrm{d}u=6x\,\mathrm{d}x$ and $\frac{1}{3}\mathrm{d}u=2x\,\mathrm{d}x$

Substituting gives $\int\frac{2x}{\sqrt{3x^2+1}}\,\mathrm{d}x=\frac{1}{3}\int\frac{1}{\sqrt{u}}\,\mathrm{d}u=\frac{1}{3}\int u^{-\frac{1}{2}}\mathrm{d}u=\frac{2}{3}u^{\frac{1}{2}}$

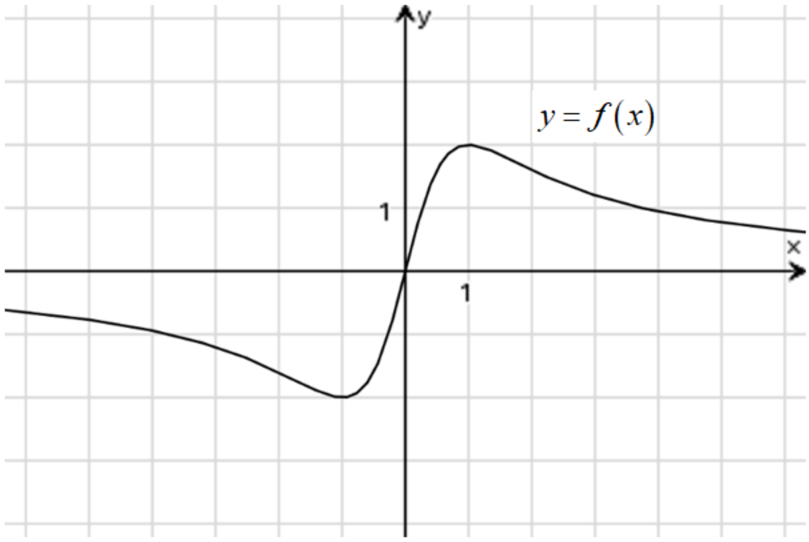
$$=\frac{2}{3}\sqrt{3x^2+1}+C$$

$g(1)=2$: $\frac{2}{3}\sqrt{4}+C=2 \;\Rightarrow\; C=2-\frac{4}{3}=\frac{2}{3}$

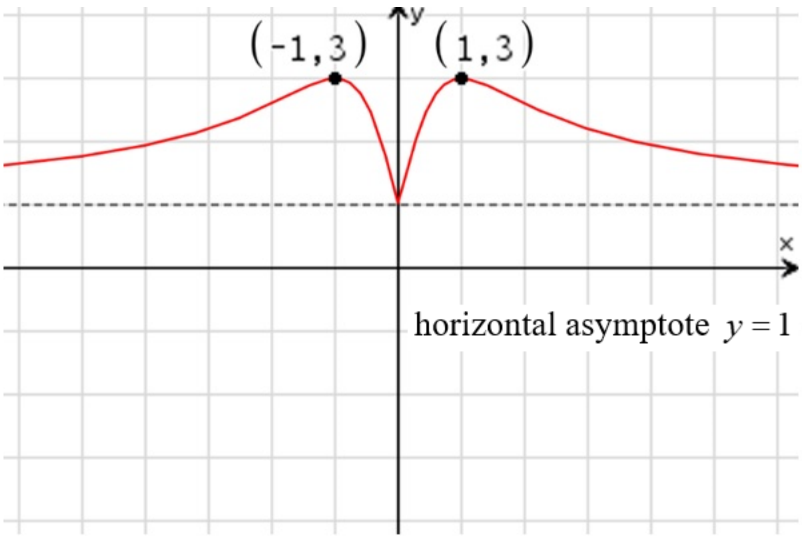
Thus, $g(x)=\frac{2}{3}\sqrt{3x^2+1}+\frac{2}{3}$

5. [Maximum mark: 5]

The diagram below shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = 0$ (x-axis) and has a minimum at $(-1, -2)$ and a maximum at $(1, 2)$.



On the set of axes below, sketch the graph of $y = f(|x|) + 1$. Clearly show any asymptotes with their equations and the coordinates of any maxima or minima.



6. [Maximum mark: 6]

A geometric series has a common ratio of 2^x.

(a) Find the values of x for which the sum to infinity of the series exists. [2]

(b) If the first term of the series is 14 and the sum to infinity is 16, find the value of x. [4]

(a) S_∞ exists if $-1 < r < 1 \Rightarrow |r| < 1$

$|2^x| < 1 \Rightarrow x < 0$

(b) $S_\infty = \dfrac{u_1}{1-r}$

$16 = \dfrac{14}{1-2^x}$

$1 - 2^x = \dfrac{14}{16} = \dfrac{7}{8}$

$2^x = \dfrac{1}{8}$

$x = -3$

7. [Maximum mark: 6]

Consider the curve with the equation $x^2 - xy + y^2 = 6$.

(a) State the coordinates of all the points where the curve intersects the *x*-axis. [2]

(b) Find the equation for each of the two vertical lines that are tangent to the curve. [4]

(a) curve intersects *x*-axis when $y = 0$

$$x^2 - x(0) + (0)^2 = 6 \Rightarrow x^2 = 6 \Rightarrow x = \pm\sqrt{6}$$

Thus, the curve intersects the *x*-axis at $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$.

(b) $\dfrac{d}{dx}\left(x^2 - xy + y^2\right) = \dfrac{d}{dx}(6) \Rightarrow 2x - \left(y + x\dfrac{dy}{dx}\right) + 2y\dfrac{dy}{dx} = 0$

$$x\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - y \Rightarrow \frac{dy}{dx} = \frac{2x - y}{x - 2y}$$

The gradient of a vertical line is undefined.

$\dfrac{dy}{dx}$ is undefined when $x - 2y = 0 \Rightarrow y = \dfrac{x}{2}$

Substituting gives $x^2 - x \cdot \dfrac{x}{2} + \left(\dfrac{x}{2}\right)^2 = 6 \Rightarrow \dfrac{3x^2}{4} = 6 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$

Thus, the equations of the two vertical tangent lines are $x = -2\sqrt{2}$ and $x = 2\sqrt{2}$

[or $x = -\sqrt{8}$ and $x = \sqrt{8}$]

8. [Maximum mark: 8]

The equation $4x^2+3x+2=0$ has roots α and β.

(a) Write down the values of $\alpha+\beta$ and $\alpha\beta$. [2]

(b) Hence, show that $\alpha^2+\beta^2=-\dfrac{7}{16}$. [2]

(c) Hence, Find an equation with integer coefficients that has roots $2\alpha-\beta$ and $2\beta-\alpha$. [4]

(a) $\alpha+\beta=-\dfrac{3}{4}$

$\alpha\beta=\dfrac{1}{2}$

(b) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$=\left(-\frac{3}{4}\right)^2-2\left(\frac{1}{2}\right)$$

$$=-\frac{7}{16}$$

(c) $\left[x-(2\alpha-\beta)\right]\left[x-(2\beta-\alpha)\right]=0$

$$x^2-\left((2\alpha-\beta)+(2\beta-\alpha)\right)x+(2\alpha-\beta)(2\beta-\alpha)=0$$

$$x^2-(\alpha+\beta)x+5\alpha\beta-2\left(\alpha^2+\beta^2\right)=0$$

$$x^2+\frac{3}{4}x+\frac{5}{2}+\frac{7}{8}=0$$

$$x^2+\frac{3}{4}x+\frac{27}{8}=0$$

$$8x^2+6x+27=0$$

9. [Maximum mark: 8]

Using mathematical induction, prove that $n(n+1)(n+2)$ is divisible by 6 for all n, $n \in \mathbb{Z}^+$.

(i) Show statement true for $n=1$:

$n(n+1)(n+2)=1(1+1)(1+2)=6$ and 6 is divisible by 6

Thus, statement is true for $n=1$.

(ii) Assume statement is true for a specific value of $n \in \mathbb{Z}^+$; i.e. true for some $n=k$:

that is, assume $k(k+1)(k+2)$ is divisible by 6; which means that $k(k+1)(k+2)$ must be a multiple of 6; i.e. $k(k+1)(k+2)=6p$ where $p \in \mathbb{Z}^+$

(iii) Show that it must follow from this assumption that the statement is true for the next value of *n*; that is, show statement must be true for $n=k+1$:

need to show that $(k+1)(k+2)(k+3)$ is a multiple of 6

$$k(k+1)(k+2)+3(k+1)(k+2)$$

$$6p+3(k+1)(k+2)$$ [applying assumption from (ii)]

$(k+1)(k+2)$ is a product of two consecutive integers – thus, one factor is even (a multiple of 2);

Therefore, the product $(k+1)(k+2)$ must be a multiple of 2;

That is, $(k+1)(k+2)=2q,\ q \in \mathbb{Z}^+$

$$6p+3(k+1)(k+2)=6p+3\cdot 2q=6p+6q=6(p+q)$$

Thus, it follows that $(k+1)(k+2)(k+3)$ is divisible by 6. ***Q.E.D.***

The statement has been shown true for $n=1$, and given true for some $n=k, n \in \mathbb{Z}^+$ it follows that it must also be true for $n=k+1$; therefore, by the principle of mathematical induction the statement is true for all $n \in \mathbb{Z}^+$.

Section B

10. [Maximum mark: 16]

The function f is defined as $f(x)=\dfrac{x+1}{\ln(x+1)}$, $x>0$.

(a) (i) Show that $f'(x)=\dfrac{\ln(x+1)-1}{(\ln(x+1))^2}$.

(ii) Find $f''(x)$, writing it as a single rational expression [6]

(b) (i) Find the value of x satisfying the equation $f'(x)=0$.

(ii) Show that this value gives a minimum value for $f(x)$, and determine the minimum value of the function. [7]

(c) Find the x-coordinate of the one point of inflexion on the graph of f. [3]

■ **worked solution** ■

(a) (i) Using the quotient rule.

$$f'(x)=\frac{(\ln(x+1))(1)-(x+1)\left(\dfrac{1}{x+1}\right)}{(\ln(x+1))^2}$$

So $f'(x)=\dfrac{\ln(x+1)-1}{(\ln(x+1))^2}$

(ii) **METHOD 1**

Using the quotient rule.

$$f''(x)=\frac{\dfrac{(\ln(x+1))^2}{x+1}-\dfrac{2\ln(x+1)(\ln(x+1)-1)}{x+1}}{(\ln(x+1))^4}$$

$$=\frac{2\ln(x+1)-(\ln(x+1))^2}{(x+1)(\ln(x+1))^4}$$

$$=\frac{2-\ln(x+1)}{(x+1)(\ln(x+1))^3}$$

worked solution for question 10 continues on next page >>

worked solution for question 10 continued

(a) (ii) **METHOD 2**

$$f'(x)=\frac{1}{\ln(x+1)}-\frac{1}{\left(\ln(x+1)\right)^2}$$

$$f''(x)=\frac{-1}{(x+1)\left(\ln(x+1)\right)^2}+\frac{2}{(x+1)\left(\ln(x+1)\right)^3}$$

$$=\frac{2-\ln(x+1)}{(x+1)\left(\ln(x+1)\right)^3}$$

(b) (i) $\ln(x+1)=1$

$x=\mathrm{e}-1$

(ii) **METHOD 1**

Using a first derivative test.

For example, when $x=1$, $f'(x)=\ln 2-1(<0)$.

For example, when $x=2$, $f'(x)=\ln 3-1(>0)$.

Hence, $x=\mathrm{e}-1$ gives a minimum value.

$$f(\mathrm{e}-1)=\frac{\mathrm{e}-1+1}{\ln(\mathrm{e}-1+1)}=\frac{\mathrm{e}}{\ln(\mathrm{e})}=\frac{\mathrm{e}}{1}=\mathrm{e}$$

Thus, the minimum value is e.

METHOD 2

Using the second derivative test.

$$f''(\mathrm{e}-1)=\frac{1}{\mathrm{e}}>0$$

Hence, $x=\mathrm{e}-1$ gives a minimum value.

$$f(\mathrm{e}-1)=\frac{\mathrm{e}-1+1}{\ln(\mathrm{e}-1+1)}=\frac{\mathrm{e}}{\ln(\mathrm{e})}=\frac{\mathrm{e}}{1}=\mathrm{e}$$

Thus, the minimum value is e.

(c) $2-\ln(x+1)=0$

$\ln(x+1)=2$

$x=\mathrm{e}^2-1$

11. [Maximum mark: 20]

The points A, B and C have position vectors $\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$, $3\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$ and $-\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$ respectively and lie in plane Π.

(a) Find: (i) the area of triangle ABC;

(ii) the shortest distance from C to the line AB.

(iii) a Cartesian equation of plane Π. [14]

The line L passes through the origin and is normal to plane Π and L intersects Π at point D.

(b) Find: (i) the coordinates of point D;

(ii) the distance of Π from the origin. [6]

■ **worked solution** ■

(a) (i) **METHOD 1**

$$\vec{AB}=\begin{pmatrix}3\\1\\0\end{pmatrix}-\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}2\\-1\\1\end{pmatrix}\qquad \vec{AC}=\begin{pmatrix}-1\\1\\2\end{pmatrix}-\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}-2\\-1\\1\end{pmatrix}$$

$$\vec{AB}\times\vec{AC}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & -1 & 1\\ -2 & -1 & 1\end{vmatrix}=\boldsymbol{i}(-1+1)-\boldsymbol{j}(2+2)+\boldsymbol{k}(-2-2)=-4\boldsymbol{j}-4\boldsymbol{k}$$

$$\text{Area of triangle ABC }=\frac{1}{2}\left|-4\boldsymbol{j}-4\boldsymbol{k}\right|=\frac{1}{2}\sqrt{32}\ \text{ or }\ 2\sqrt{2}\ \text{ square units}$$

METHOD 2

$$|\text{AB}|=\sqrt{6},\ \ |\text{BC}|=4,\ \ |\text{AC}|=\sqrt{6}$$

$$\text{using cosine rule: } \cos A=\frac{6+6-16}{2\cdot 6}=-\frac{1}{3}\ \Rightarrow\ A=\arccos\left(-\frac{1}{3}\right)$$

$$\text{Area of triangle ABC }=\frac{1}{2}bc\sin A=\frac{1}{2}\sqrt{6}\sqrt{6}\sin\left(\arccos\left(-\frac{1}{3}\right)\right)$$

$$=3\sin\left(\arccos\left(-\frac{1}{3}\right)\right)=3\left(\frac{2\sqrt{2}}{3}\right)=2\sqrt{2}\ \text{ square units}$$

(ii) $\text{Area}=2\sqrt{2}=\frac{1}{2}\text{AB}\times h=\frac{1}{2}h\sqrt{6}$ where h is shortest distance from C to AB (height of $\triangle$)

$$h=\frac{2\sqrt{2}}{\frac{1}{2}\sqrt{6}}=4\sqrt{\frac{1}{3}}=\frac{4\sqrt{3}}{3}\ \text{ units}$$

worked solution for question 11 continues on next page >>

worked solution for question 11 continued

(a) (iii) Cartesian equation of plane $\prod$ has the form $ax+by+cz=d$

From the working in part (i), known that normal vector for plane $\prod$ is $\begin{pmatrix}0\\-4\\-4\end{pmatrix}$, or $\begin{pmatrix}0\\1\\1\end{pmatrix}$

Since $(1,2,1)$ is on the plane, then $0(1)+1(2)+1(1)=d \;\Rightarrow\; d=3$

Thus, a Cartesian equation of plane $\prod$ is $y+z=3$

(b) (i) The equation of OD is $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\1\end{pmatrix} \;\Rightarrow\; \begin{pmatrix}x\\y\\z\end{pmatrix}=\lambda\begin{pmatrix}0\\1\\1\end{pmatrix} \;\Rightarrow\; \begin{cases}x=0\\y=\lambda\\z=\lambda\end{cases}$

Substituting into Cartesian equation of plane $\prod$ gives $\lambda+\lambda=3 \;\Rightarrow\; \lambda=\frac{3}{2}$

Thus, coordinates of D are $\left(0,\frac{3}{2},\frac{3}{2}\right)$

(ii) The length of $\overrightarrow{\text{OD}}$ is the distance from plane $\prod$ to the origin.

$$\left|\overrightarrow{\text{OD}}\right|=\sqrt{0+\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2}=\sqrt{\frac{9}{2}}=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}\text{ units}$$

12. [Maximum mark: 18]

(a) Find the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^4$ and write it in the form $a + b\mathrm{i}$, where a and b are in terms of $\sin\theta$ and $\cos\theta$. [4]

(b) Hence, using De Moivre's theorem, show that $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$. [3]

(c) Hence, show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$. [5]

(d) Hence, find the four solutions to $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$. [6]

■ **worked solution** ■

(a) Applying binomial theorem:

$$(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta(\mathrm{i}\sin\theta) + 6\cos^2\theta(\mathrm{i}\sin\theta)^2 + 4\cos\theta(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4$$

$$= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta$$

Thus, $(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + \mathrm{i}\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$

(b) Using de Moivre's theorem: $(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$

Hence, $\cos 4\theta + \mathrm{i}\sin 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + \mathrm{i}\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$

Equating real parts gives: $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ ***Q.E.D.***

(c) Considering $\cos 4\theta + \mathrm{i}\sin 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + \mathrm{i}\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$

Equating imaginary parts gives: $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

Hence, $\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta} = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

Multiplying by $\dfrac{\sin^4\theta}{\cos^4\theta}$: $= \dfrac{\dfrac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \dfrac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\dfrac{\cos^4\theta}{\cos^4\theta} - \dfrac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \dfrac{\sin^4\theta}{\cos^4\theta}}$

$$= \frac{\dfrac{4\cancel{\cos^3\theta}}{\cancel{\cos^3\theta}}\cdot\dfrac{\sin\theta}{\cos\theta} - \dfrac{4\cancel{\cos\theta}}{\cancel{\cos\theta}}\cdot\dfrac{\sin^3\theta}{\cos^3\theta}}{\dfrac{\cancel{\cos^4\theta}}{\cancel{\cos^4\theta}} - \dfrac{6\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}}\cdot\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\sin^4\theta}{\cos^4\theta}}$$

Thus, $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ ***Q.E.D.***

worked solution for question 12 continues on next page >>

worked solution for question 12 continued

(d) $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$x^4 - 6x^2 + 1 = 4x - 4x^3$

$\dfrac{x^4 - 6x^2 + 1}{x^4 - 6x^2 + 1} = \dfrac{4x - 4x^3}{x^4 - 6x^2 + 1}$

$1 = \dfrac{4x - 4x^3}{x^4 - 6x^2 + 1}$

Let $x = \tan\theta$: $1 = \dfrac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1}$

Hence, $\tan 4\theta = 1 \;\Rightarrow\; 4\theta = \dfrac{\pi}{4} + k\pi,\ k \in \mathbb{Z} \;\Rightarrow\; \theta = \dfrac{\pi}{16} + k\dfrac{\pi}{4},\ k \in \mathbb{Z}$

For $k = 0, 1, 2, 3$: $\theta = \dfrac{\pi}{16}, \dfrac{5\pi}{16}, \dfrac{9\pi}{16}, \dfrac{13\pi}{16}$

Because period of tangent is π, other values of k will repeat solutions for θ

Thus, $x = \tan\dfrac{\pi}{16}, \tan\dfrac{5\pi}{16}, \tan\dfrac{9\pi}{16}, \tan\dfrac{13\pi}{16}$

Note: other sets of four solutions allowed; e.g. $x = \tan\dfrac{5\pi}{16}, \tan\dfrac{9\pi}{16}, \tan\dfrac{13\pi}{16}, \tan\dfrac{17\pi}{16}$